ON EXTREMUM PROPERTIES OF PLASTICITY CONDITIONS

(OB EKSTREMAL'NYKH SVOISTVAKH USLOVII PLASTICHNOSTI)

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D. D. IVLEV
(Voronezh)

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1. The theory of isotropic rigid-plastic body is the most developed chapter of the mathematical theory of plasticity [1-6]. A characteristic feature of the ideally plastic flow is its shear mechanism and, consequently, an appearance of slip lines and slip surfaces. The mathematical formulation of these phenomena reduces to hyperbolic differential equations.

In [7] was considered the problem of the construction of a theory for a body which is: a) homogeneous, b) ideal, c) isotropic, d) incompressible, e) rigid-plastic and f) which exhibits like behavior in tension and compression, i.e. the yield points for tension and compression are equal, and, with the reversing of the stress sign, the velocities also reverse their sign. We shall call in the sequel the isotropic bodies which satisfy condition (f) normal isotropic bodies (normal isotropy); otherwise, we shall call them anormally isotropic bodies (anormal isotropy).

The following proposition was advanced [1-6]: among all admissible plasticity conditions within a given group of mechanical properties of materials there exists only one true condition. We note also that a given plasticity condition determines a plastic flow law associated with it.

It was also shown that a true plasticity condition for a homogeneous, ideal, normally isotropic, rigid-plastic body is the Tresca condition.

It is known [8,9] that the Tresca condition permits a theory to be developed for an ideally plastic body with unique qualitative characteristics, which correspond well to the shear behavior of ideally plastic flow.

It is also known that any other plasticity condition, except for the cases when such a condition essentially coincides with the Tresca condition, leads to equations which in the general case give results which do not correspond to the qualitative nature of ideal plastic flow. Among others, the von Mises condition belongs to this group, and it leads in the general case to elliptic differential equations.

The class of all admissible plasticity conditions is determined by a family of functions convex relative to the origin:

$$f(\Sigma_2, |\Sigma_3|) = \text{const} \tag{1.1}$$

where Σ_2, Σ_3 are the second and third invariants of the stress deviator tensor respectively.

For torsion and plane-strain problems (and for some other cases) all plasticity conditions are reduced to one, namely

$$\Sigma_2 = \text{const} \tag{1.2}$$

For plane-stress and three-dimensional general axisymmetric problems there exists *a priori* a possibility of choice of various plasticity conditions within the class (1.1).

In [7] it was assumed that the value of the yield stress in tension and compression for all plasticity conditions (1.1) is the same. Thus, in each specific case an arbitrary constant was detèrmined, which appears on the right-hand side of (1.1), and thus a class of all admissible plasticity conditions has been determined (Fig. 1).

Obviously, if an additional requirement is set forth that a class of admissible plasticity conditions satisfies some other given stress combination, then the relative position of yield surfaces will be different [10] (Fig. 2). In the case, when for a class of admissible plasticity conditions a value of the yield stress in shear is fixed, then the relative position of yield surfaces is as shown in Fig. 3.

For all considered cases of admissible plasticity conditions only one experimental point is common. This fact is explained as follows. For a given ideally plastic body the initiation of yield for a given stress combination must be completely determined. However, this condition, viz. the specification of one experimental point, does not impose any limitations on a class of admissible plasticity conditions. The selection of an experimental point determines only relative positions of the yield surfaces.

In [7] three theorems have been formulated.

Local theorem. For all given deformation increments of an element of a body the work of external forces attains its minimum for the true plasticity condition.

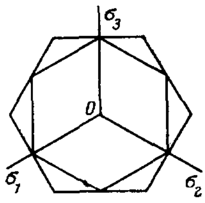

Fig. 1.

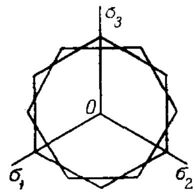

Fig. 2.

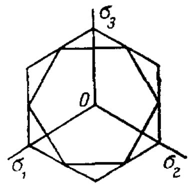

Fig. 3.

Integral theorem. (I) For given boundary displacements of a body the work of external forces attains its minimum for the true plasticity condition among all admissible plasticity conditions. (II) The work of given external forces attains its maximum for the true plasticity condition among all admissible plasticity conditions.

These theorems were formulated for completely determined configurations of admissible yield surfaces (Fig. 1). If an experimental point in shear is given (Fig. 3), then the word "minimum" has to be replaced by "maximum", and vice versa.

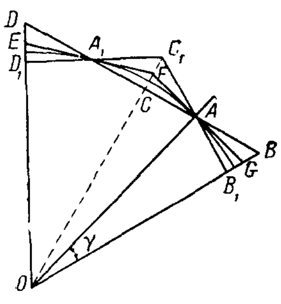

Fig. 4.

Let us consider in greater detail the case when one experimental point is given. Assume that on the basis of an experiment it was established that plastic flow of an ideal, normally isotropic, incompressible, rigid-plastic body takes place after some combination of the stresses achieves the value represented by point A in Fig. 4. Figure 4 represents one sixth of a deviatoric plane; this is sufficient for further considerations. Assume that $OA = 1$. All admissible convex yield polygons are bounded by broken lines $BCD\ldots$ and $B_1C_1D_1\ldots$. Assume now that an angle AOB is fixed and is equal to γ. A variable angle θ will be measured counter-clockwise from OB. Note that angle $BOD = 1/3\ \pi$ and angle $BOC = 1/6\ \pi$. We shall assume that the displacement increment vector $d\epsilon$ is given and for simplicity we set $|\ d\epsilon\ | = 1$.

To begin with, let us consider the Tresca plasticity condition represented by the segment BD. The direction of flow for the interior points of BD is $\theta = 1/6\ \pi$, and the other remaining flow directions correspond to the points B and D.

It is easy to see that the magnitude of the work of the stresses along given deformation increments for the Tresca plasticity condition will have the form

$$dA_T = \sigma d\varepsilon = \frac{2\sqrt{3}}{3}\cos\left(\frac{1}{6}\pi - \gamma\right)\cos\theta \tag{1.3}$$

To the segment B_1C_1 corresponds the flow direction $\theta = 0$, and to the segment C_1D_1 corresponds the flow direction $\theta = 1/3\,\pi$. The other remaining directions correspond to the vertex C_1. Clearly, the magnitude of the work for this plasticity condition is

$$dA^* = \sigma d\varepsilon = \frac{2\sqrt{3}}{3}\cos\gamma\cos\left(\frac{1}{6}\pi - \theta\right) \tag{1.4}$$

The von Mises plasticity condition is characterized by the magnitude of the work

$$dA_M = \sigma d\varepsilon = 1 \tag{1.5}$$

Hencky [11] established the stationary character of the work for the case of the von Mises plasticity condition.

Figure 5 represents the variation of the work with angle θ.

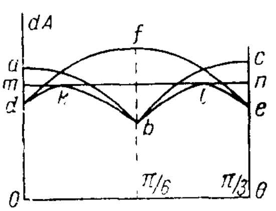

Fig. 5.

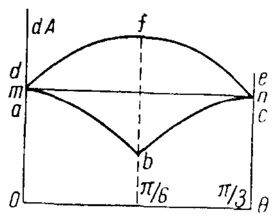

Fig. 6.

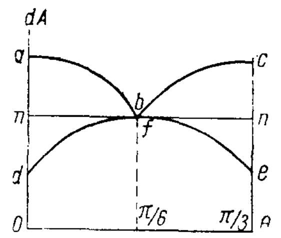

Fig. 7.

Furthermore, the variation of dA_T is represented by a curve *abc* and that of dA^* by *dfe*. The horizontal line *mn* corresponds to the work dA_M for the von Mises plasticity condition.

Obviously, the plasticity condition satisfying the minimum work requirement is represented by the broken line B_1ACA_1D. In this case the work is

$$\begin{aligned} dA_1 &= \cos(\theta - \gamma) && (0 \leqslant \theta \leqslant {}^1/_6\,\pi) \\ dA_1 &= \cos({}^1/_3\,\pi - \theta - \gamma) && ({}^1/_6\,\pi \leqslant \theta \leqslant {}^1/_3\,\pi) \end{aligned} \tag{1.6}$$

A corresponding curve in Fig. 5 is *dkble*. If $\gamma = 0$, i.e. if the original experiment is simple tension or compression, then the work diagram is as shown in Fig. 6. In this case the Tresca plasticity condition satisfies the minimum work requirement [7].

If $\gamma = 1/6\ \pi$, i.e. if the original experiment is pure shear ($\sigma_1 = -\sigma_2$, $2\ \sigma_3 - \sigma_1 - \sigma_2 = 0$), then the Tresca plasticity condition satisfies the maximum work requirement among all other admissible plasticity conditions (Fig. 7).

The von Mises plasticity condition does not satisfy the extremum condition of work along given displacement increments among all admissible plasticity conditions. The von Mises circle is bounded below by a broken line $B_1ACA_1D_1\ldots$, and from above by a broken line $EFG\ldots$ (Fig. 4). In other words, for an arbitrary original experimental point and for arbitrary deformation increments it is possible to find within a class of admissible plasticity conditions two such plasticity conditions for which

$$dA_1 \leqslant dA_M \leqslant dA_2 \tag{1.7}$$

where dA_1 and dA_2 are the increments of work of the stresses for some specially selected plasticity conditions.

We shall now formulate the above point of view. In the theory of a homogeneous, ideal, incompressible, normally isotropic, rigid-plastic body there exists only one true plasticity condition (a basic proposition).

There are evidences (such as qualitative behavior of observed flows of metals being close to the ideally plastic behavior; the experiments which demonstrate that the metals with the more pronounced plateau at the yield point behave in closer conformity with the Tresca plasticity condition) which permit the assertion that this true plasticity condition is the Tresca condition.

There exist energy criteria which single out the Tresca plasticity condition among the class of all admissible conditions, if one accepts as the criterion of the class of admissible conditions a known value of initiation of yield in tension, compression or pure shear.

The development of such concepts may permit us to formulate a criterion for the determination of a true relationship between the stresses and deformations within a given group of the mechanical properties of materials.

We would like to dwell longer on the work [10]. In the first section of this work one finds the following: "close attention was given to the determination of a plasticity condition for the case when a limiting tensile stress is given and when the two extremal principles are utilized".

For a body element in [7] an extremum of the work of the stresses along given increments of deformations was determined; in other words, a vector $d\epsilon$ was given. (The modulus of this vector, for the sake of simplicity, was taken as unity). Then the following expression was considered:

$$dA = \boldsymbol{\sigma} d\boldsymbol{\varepsilon} = \sigma \cos(\boldsymbol{\sigma}, d\boldsymbol{\varepsilon}) \tag{2.1}$$

In [10] two extremal principles are mentioned for a given element of a body. One does not find, however, any formulation of these principles there, and moreover in both cases an extremum of the same quantity (2.1) was considered.

When an element of a body is considered one can assign a vector $d\epsilon$ and compare the magnitude of the work of the stresses (2.1) along given deformation increments for all admissible plasticity conditions. It is impossible, however, to assign in a general case the stress-vector $\boldsymbol{\sigma}$, and thus it is meaningless to speak about two local variational principles.

In the first section of [10] it is pointed out that if the yield surface of an admissible plasticity condition lies outside the von Mises circle, then the work (2.1) is greater or equal to the same expression for the von Mises circle and, conversely, if the yield surface of an admissible plasticity condition lies inside the von Mises circle then the work (2.1) is less or equal to the same for the von Mises circle.

But this is a special case of a more general situation.* If in the deviatoric plane a yield surface of some admissible plasticity condition is outside another yield surface (they may have common points), then the work (2.1) for the former is larger or equal to the work for the latter. Note that Expressions (2.1) are equal for a given vector $d\epsilon$.

In the second section of [10] it is said: "D.D. Ivlev points out in his work that he does not take into consideration the other experimental points because of the inaccuracy of the experimental procedures".

A simple review of [7] would show that no such statement was made.

The problem is not that of accuracy of the experiments. An experiment may be executed perfectly. A given experiment, however, is dealing with real materials and the ideally plastic body is only a model. Thus, a more accurate experiment will show more drastically the deviations in the

* Iu.N. Rabotnov drew the author's attention to this fact in 1958 while [7] was in progress.

behavior of real materials and models.

The use of one experimental point, as has been demonstrated, does not impose any limitations on the class of admissible plasticity conditions, and permits the consideration of the totality of all admissible plasticity conditions.

Next in [10] it was said: "Since from the point of view of the simplest isotropic, ideally plastic theory the selection of a fixed experimental point should not influence the results of the investigations, it can be asserted that in the most general case of specification of an experimental point only the von Mises plasticity condition remains a physical invariant characteristic."

To date, the ideally plastic theory has dealt with the concept of an isotropic body but not with the concept of "an isotropic theory" (this terminology appears twice in [10]). This terminology should be clarified.

If the von Mises plasticity condition were only a physical invariant characteristic, then the question would automatically be resolved in favor of the author of [10]. It is not so, however, and this has been known since the time of Tresca. Moreover, any plasticity condition formulated in terms of the invariants is an invariant characteristic.

In the framework of the considered energy criteria there does not exist a single case where the von Mises plasticity condition appears to be preferable to any other condition.

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